

Vanishing moment conditions for wavelet atoms in higher dimensions

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Abstract

We provide explicit criteria for wavelets to give rise to frames and atomic decompositions in $L^2(\mathbb{R}^d)$, but also in more general Banach function spaces. We consider wavelet systems that arise by translating and dilating the mother wavelet, with the dilations taken from a suitable subgroup of $GL(\mathbb{R}^d)$, the so-called *dilation group*. The paper provides a unified approach that is applicable to a wide range of dilation groups, thus giving rise to new atomic decompositions for homogeneous Besov spaces in arbitrary dimensions, but also for other function spaces such as shearlet coorbit spaces.

The atomic decomposition results are obtained by applying the coorbit theory developed by Feichtinger and Gröchenig, and they can be informally described as follows: Given a function $\psi \in L^2(\mathbb{R}^d)$ satisfying fairly mild decay, smoothness and vanishing moment conditions, *any* sufficiently fine sampling of the translations and dilations will give rise to a wavelet frame. Furthermore, the containment of the analyzed signal in certain smoothness spaces (generalizing the homogeneous Besov spaces) can be decided by looking at the frame coefficients, and convergence of the frame expansion holds in the norms of these spaces. We motivate these results by discussing nonlinear approximation.

Keywords: square-integrable group representation; continuous wavelet transform; coorbit spaces; Banach frames; irregular wavelet frames; vanishing moments; nonlinear approximation; shearlets; anisotropic wavelet systems; nonlinear approximation

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1 Introduction

The great success of wavelet theory in applications largely depends on two features: Approximation-theoretic properties of wavelet orthonormal bases, and the availability of fast algorithms in the discrete-domain setting. There are many facets to the approximation-theoretic properties; in this exposition I will focus mainly on a particularly appealing set of results, namely the wavelet characterization of homogeneous Besov spaces. Given a wavelet orthonormal basis $(\psi_{j,k})_{j,k \in \mathbb{Z}} \subset L^2(\mathbb{R})$, every function $f \in L^2(\mathbb{R})$ has the expansion

$$f = \sum_{j,k} \langle f, \psi_{j,k} \rangle \psi_{j,k} , \quad (1)$$

with a square-summable coefficient family $(\langle f, \psi_{j,k} \rangle)_{j,k \in \mathbb{Z}}$. More importantly however, (1) converges in smoothness spaces such as the homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R})$, as soon as f belongs

to that space. In fact, the wavelet system is an unconditional basis of that space, and membership of $f \in \dot{B}_{p,q}^s(\mathbb{R})$ is equivalent to a suitable weighted $\ell^{p,q}$ -summability condition on the coefficients (thus can be decided just by looking at the coefficients). Thus the wavelet decomposition is valid *simultaneously* in a whole range of smoothness spaces, and this observation provides a solid theoretic foundation for the development and study of algorithms for applications like denoising or compression. (See subsection 1.1 for a discussion of nonlinear approximation.)

In this paper, we wish to extend these results to arbitrary dimensions, replacing dyadic dilations in dimension one by certain rather general groups H of (typically nonscalar) dilations; H is called the *dilation group*. The rationale for choosing H is explained in more detail in the next section. This is a rather general setting with a lot of different groups to choose from, including the similitude group in arbitrary dimensions, diagonal groups, but also the shearlet group that has received considerable attention in the past few years; see [20] for an introduction.

For arbitrary dilation groups, the existence of associated orthonormal wavelet bases is not established, and we will thus be concerned with frame rather than ONB expansions. The frame expansions will be obtained as discretization of the *continuous wavelet transform* associated to the matrix group H , by applying the coorbit theory established by Feichtinger and Gröchenig, see [7, 8, 9, 17]. The previous paper [16] established that coorbit theory applies to a large class of dilation groups and their associated continuous wavelet transforms, and thus provides the existence of a consistently defined scale of Besov-type function spaces, atomic decompositions in terms of bandlimited Schwartz functions, etc. Furthermore, [16] developed vanishing moment criteria for analyzing windows, which mimic the simple criteria for wavelet ONB's mentioned above.

In principle, coorbit theory provides a description of “nice” wavelets that can be used to obtain simultaneous atomic decompositions for a whole range of Besov-type spaces; this was used in [16] to show the existence of band-limited atoms for this setting. However, both for practical and theoretical considerations, one would also like to replace bandlimited atoms by, say, compactly supported ones. This raises the challenge of finding explicit and easily fulfilled criteria for “nice” wavelets with compact support (or, more generally, with certain decay and/or integrability conditions), and it is the chief purpose to provide such criteria for general dilation groups.

As a result, we will obtain a very general approach to the construction of wavelet frames in higher dimensions, with very mild conditions on the wavelets to be chosen, and a large class of dilation groups to choose from. The latter aspect is of particular relevance for the ongoing search for anisotropic wavelet systems designed to resolve singularities in dimensions two and higher, such as the shearlet systems.

While the construction of wavelet frames and bases is by now very well understood (at least for some groups), constructing such systems, and guaranteeing their properties, is still fairly cumbersome. ONB's are typically constructed from a multiresolution analysis, and their existence has been established (to my present knowledge) only for very few classes of dilations, such as dyadic dilations. In the construction of frames, the sampling set used to discretize shifts and dilations is typically fixed beforehand, and then the frame (or ONB) generators are constructed for this particular choice of sampling set, see [2] for the prototype. Furthermore, the constructions for concrete groups, as in the shearlet case, are typically tailored to the specific structure of the dilation group. By contrast, the discretization methods developed in [7, 8, 9] start from a given “nice” wavelet and yield that the action of *any* sufficiently dense uniformly discrete sampling set contained in the underlying affine group will give rise to a frame. The price one pays for this generality lies in the absence of explicit sampling densities and frame bounds. It

is the chief contribution of this paper to provide explicit and easily verifiable criteria that allow to spot a nice wavelet.

1.1 Nonlinear approximation using wavelet frames

To illustrate the results in our paper, let us take a closer look at the problem of nonlinear approximation. The following discussion loosely follows [3, Section 4.4]. Let $(\psi_\lambda)_{\lambda \in \Lambda} \subset \mathcal{H}$ denote any system of vectors in a Hilbert space \mathcal{H} . We define the associated nonlinear approximation error by

$$E_n(f; (\psi_\lambda)_\lambda) = \inf \left\{ \left\| f - \sum_{\lambda \in \Lambda'} c_\lambda \psi_\lambda \right\|_2 : c_\lambda \in \mathbb{C}, |\Lambda'| \leq n \right\}. \quad (2)$$

A famous result in wavelet approximation theory states that the nonlinear approximation error of $f \in L^2(\mathbb{R})$ in a wavelet orthonormal basis $(\psi_{j,k})_{j,k \in \mathbb{Z}} \subset L^2(\mathbb{R})$ fulfills

$$\sum_{n=1}^{\infty} n^{-p/2} E_n(f; (\psi_{j,k}))^p < \infty$$

for some $0 \leq p < 2$ iff $f \in \dot{B}_{p,p}^{1/p-1/2}(\mathbb{R})$; see e.g. [6] for a much more complete discussion. Note that finiteness of the sum can be understood as a sort of decay condition on the approximation error, which becomes sharper as p decreases.

There are however some conditions that a wavelet has to meet: The precise range of homogeneous Besov spaces for which the above equivalence is valid depends on properties of the wavelet, typically formulated in terms of decay, smoothness and vanishing moments. However, it should be stressed that these conditions are fairly easy to verify, and the existence of wavelet ONB's fulfilling them to any prescribed (finite) order has been established early on.

If one replaces bases by frames, the above sharp characterization of nonlinear approximation rate and p -summability of the coefficients no longer holds; however, at least one direction remains intact. The proof of the following proposition follows by the exact same reasoning as in [3, Theorem 4.9].

Proposition 1.1. *Let $(\psi_\lambda)_{\lambda \in \Lambda}$ denote a frame in the Hilbert space \mathcal{H} . Given $f \in \mathcal{H}$, let*

$$f = \sum_{\lambda \in \Lambda} c_\lambda f(\lambda)$$

with suitable coefficients $(c_\lambda)_{\lambda \in \Lambda}$. Let $1 \leq p \leq 2$. Then there exists a constant C depending on p and the frame constants such that if $(c_\lambda)_{\lambda \in \Lambda} \in \ell^p(\Lambda)$, one has

$$\left(\sum_{n=1}^{\infty} n^{-p/2} E_n(f; (\psi_{j,k}))^p \right)^{1/p} \leq C \|(c_\lambda)_\lambda\|_p.$$

Using this observation, we can formulate an application of the results in this paper to homogeneous Besov spaces. The chief difference to the existing results (as far as I know them) is the great freedom in choosing the analysing function: *Any* reasonable regular function fulfilling explicit decay and vanishing moment conditions will give rise to a wavelet frame with properties analogous to wavelet ONB's, as long as the sampling in time, scale and rotation is sufficiently fine. For the proof, we refer to Remark 4.5 below. Note that the theorem employs the usual notations $x^j = x_1^{j_1} \dots x_d^{j_d}$ for vectors $x \in \mathbb{R}^d$ and multiindices $j \in \mathbb{N}_0^d$, as well as $|j| = \sum_{i=1}^d j_i$.

Theorem 1.2. *Let $k \in \mathbb{N}$. Assume that $\psi \in L^2(\mathbb{R}^d) \cap C^{2k}(\mathbb{R}^d)$ fulfills*

$$\forall 0 \leq |j| < k : \int_{\mathbb{R}^d} x^j \psi(x) dx = 0 ,$$

with absolute convergence. Assume further that all partial derivatives of ψ of order up to $2k$ are integrable.

If $k > \frac{9}{2}d + 3$, there exist $\epsilon > 0$ and a neighborhood $U \subset SO(d)$ of the identity matrix, both depending only on ψ , such that for all $\delta_1, \delta_2 < \epsilon$ and all finite sets $\{h_1, \dots, h_r\} \subset SO(d)$ satisfying

$$SO(d) = \bigcup_{\ell=1}^r h_\ell U ,$$

the wavelet system

$$(\psi_{j,k,\ell})_{j,k \in \mathbb{Z}, \ell=1, \dots, d} \text{ with } \psi_{j,k,\ell}(t) = (1 + \delta_1)^{j/2} \psi((1 + \delta_1)^j h_\ell^{-1} t - \delta_2 k)$$

is a frame for $L^2(\mathbb{R}^d)$. Furthermore, we find for any $f \in L^2(\mathbb{R}^d)$ that $f \in \dot{B}_{p,p}^{d/2-d/p}(\mathbb{R}^d)$ iff the coefficient family $(\langle f, \psi_{j,k,\ell} \rangle)_{j,k \in \mathbb{Z}, \ell=1, \dots, d}$ is p -summable. In this case, the approximation error $E_n(f; (\psi_{j,k,\ell})_{j,k,\ell})$ fulfills

$$\left(\sum_{n=1}^{\infty} n^{-p/2} E_n(f; (\psi_{j,k,\ell})_{j,k,\ell})^p \right)^{1/p} \leq C \|f\|_{\dot{B}_{p,p}^{d/2-d/p}} .$$

Observe that is easy to construct functions ψ as in the theorem: Simply pick a function ρ with integrable derivatives of order up to $3k$, and differentiate k times. If the function ψ is isotropic, the wavelet transform is constant on $SO(d)$ -cosets, and the rotations h_1, \dots, h_r can be omitted from the theorem. Isotropic wavelets are constructed by picking a suitable isotropic function and applying the Laplacian sufficiently often.

The (relevant) case $0 < p < 1$ is excluded here, chiefly because the results established in this paper and the precursor [16] are confined to coorbit spaces associated to Banach (rather than quasi-Banach) spaces. The extension to quasi-Banach spaces is the subject of ongoing research. As far as I am aware, the theorem is new, even for dimension one. The sampling set in the theorem was chosen as regular grid mostly for the sake of notational convenience; the underlying sampling theorems due to Feichtinger and Gröchenig theory allow much more general sampling sets, see 2.7 below. For these irregular sampling sets, the Fourier techniques typically used to derive frame characterizations of homogeneous Besov spaces such as the φ -transform [11] fail, whereas an analog of Theorem 1.2 is still available.

The literature on irregular wavelet frames contains certain analogs to 1.2, usually restricted to the L^2 -case. To my knowledge, the paper [10] is closest to the aims and scope of the present article, but it treats only isotropic dilations. Another paper worthwhile mentioning in this context is [22]. By comparison to the results in those papers, the condition $k > \frac{9}{2}d + 3$ from the theorem seems quite restrictive. To some degree, this is to be expected from an approach that aims at treating large classes of dilation groups and function spaces other than L^2 in a unified perspective. As a rule, the explicit conditions on wavelets that are determined in this paper are derived as a proof of principle, and do not have any claim to optimality. We refer to Remark 4.1 for more detailed comments on this issue.

There is another direction in which the coorbit view provides a considerable extension of the known results: Note that the above statement arises from a particular choice of dilation group, namely the so-called *similitude group* $H = \mathbb{R}^+ \cdot SO(d)$. As will be demonstrated in Section 4, the results in this paper apply to a much larger variety of groups, including the shearlet dilation group studied, e.g., in [3, 4, 5, 20]. The first compactly supported shearlet frames for coorbit spaces other than $L^2(\mathbb{R}^2)$ were introduced fairly recently in [5], and their construction is considered an important step both for applied and theoretical purposes (e.g., for the derivation of trace theorems); see also [18] for the related case of cone-adapted shearlet frames. The methods developed in this paper provide an analog of Theorem 1.2 also for the shearlet setting: Here, the pertinent notion of vanishing moments is

$$0 \leq |j| < k, \forall \xi_2 \in \mathbb{R} : \int_{\mathbb{R}^2} x^j \psi(x) e^{-2\pi i \xi_2 x_2} dx = 0, \quad (3)$$

with absolute convergence of the integrals. Any function that possesses sufficiently many vanishing moments and integrable partial derivatives, under the action of a sufficiently dense sampling set, will then give rise to a frame; and it is very easy to produce compactly supported functions fulfilling these conditions. For comparison, [5, Corollary 3.3] imposes Fourier-side decay conditions that in fact follow from regularity of ψ and the vanishing moment conditions (3). Thus there is an obvious similarity between the cited result and the criteria derived in this paper; however, [5, Corollary 3.3] in addition requires compact support.

Again, a priori estimates of the decay of the nonlinear approximation error are available, where the role of the Besov spaces is taken over by the shearlet coorbit spaces $Co(L^p(G))$; this fact has already been pointed out in [3]. It turns out that imposing vanishing moments of order $k \geq 105$ (as defined by (3)), and integrable partial derivatives of to order up to $2k$ will allow to formulate a precise shearlet analog of Theorem 1.2; see Remark 4.11 below. (When pondering the fairly astronomical number of 105 vanishing moments, please recall the above disclaimer concerning optimality of the constants derived in this paper.)

1.2 A short overview of the paper

The present paper is a continuation of [16]. The chief purpose of both papers is to make certain useful but abstract notions from coorbit theory explicit for the concrete case of wavelet systems arising from the action of an affine group generated by the translations and a suitable closed group $H < GL(\mathbb{R}^d)$ of dilations. The key object that coorbit theory provides for the construction of atomic decompositions is the space $\mathcal{B}_{v_0} \subset L^2(\mathbb{R}^d)$; essentially, this is the set of “nice” wavelet for which analogs of Theorem 1.2 can be formulated. The formal definition of this space is fairly technical, see Section 2 below, and it is the chief contribution of this paper to provide concrete and easily verified sufficient criteria for elements of \mathcal{B}_{v_0} .

The paper is structured as follows: Section 2 contains the necessary notions and results concerning continuous wavelet transforms in higher dimensions. Coorbit theory is based on the theory of square-integrable irreducible representations, and for the setup of an affine group acting on $L^2(\mathbb{R}^d)$ in the natural way, it is very well understood, which dilation groups provide such representations. To any such group, there exists an associated *open dual orbit* \mathcal{O} , which will play a crucial role in the paper. Section 2 also contains the necessary ingredients of coorbit theory required, in particular the definition and basic properties of the spaces $Co(Y)$ and the spaces \mathcal{A}_{v_0} and \mathcal{B}_{v_0} of analyzing vectors and “nice” wavelets, respectively.

Section 3 contains the central result of this paper. Theorem 3.4 contains a sufficient criterion for nice wavelets in terms of vanishing moments. Here the proper notion of vanishing moments is crucial: A function has vanishing moments iff its Fourier transform vanishes to a certain order on the complement \mathcal{O}^c of the open dual orbit. The latter subset can best be understood as the “blind spot” of the wavelet transform, containing those frequencies which the wavelet transform cannot resolve well. (This set is of measure zero, hence the existence of the blind spot is no contradiction to the wavelet inversion formula.)

Theorem 3.4 does not come entirely for free: There is still one obstacle to its applicability, encapsulated in the notion of *strongly temperately embedded dual orbit*, see Definition 3.3. I therefore investigate, for various classes of groups, whether this condition is fulfilled. For all groups that were considered, including diagonal and similitude groups in arbitrary dimensions, as well as the shearlet group and a family of close relatives, the answer is yes. As a result one obtains concrete criteria which considerably generalize the known results.

2 Coorbit spaces over general dilation groups

First some notation: Given $f \in L^1(\mathbb{R}^d)$, its Fourier transform is defined as

$$\mathcal{F}(f)(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} dx ,$$

with $\langle \cdot, \cdot \rangle$ denoting the euclidean scalar product on \mathbb{R}^d . We will use the same symbol \mathcal{F} for the Fourier transform of tempered distributions. For any subspace $X \subset \mathcal{S}'(\mathbb{R}^d)$, we let $\mathcal{F}^{-1}X$ denote its inverse image under the Fourier transform.

In order to avoid cluttered notation, we will occasionally use the symbol $X \preceq Y$ between expressions X, Y involving one or more functions or vectors in \mathbb{R}^d , to indicate the existence of a constant $C > 0$, independent of the functions and vectors occurring in X and Y , such that $X \leq CY$. We let $|\cdot| : \mathbb{R}^d \rightarrow \mathbb{R}$ denote the euclidean norm. Given a matrix $h \in \mathbb{R}^{d \times d}$, the operator norm of the induced linear map $(\mathbb{R}^d, |\cdot|) \rightarrow (\mathbb{R}^d, |\cdot|)$ is denoted by $\|h\|_\infty$. By a slight abuse of notation we will also use $|\alpha| = \sum_{i=1}^d \alpha_i$ for multiindices α .

For $r, m > 0$, we let

$$|f|_{r,m} = \sup_{x \in \mathbb{R}^d, |\alpha| \leq r} (1 + |x|)^{-m} |\partial^\alpha f(x)| .$$

denote the associated Schwartz norm of a function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ with suitably many partial derivatives.

Let us now describe the necessary notions connected to continuous wavelet transforms. We fix a closed matrix group $H < \mathrm{GL}(d, \mathbb{R})$, the so-called **dilation group**, and let $G = \mathbb{R}^d \rtimes H$. This is the group of affine mappings generated by H and all translations. Elements of G are denoted by pairs $(x, h) \in \mathbb{R}^d \times H$, and the product of two group elements is given by $(x, h)(y, g) = (x + hy, hg)$. The left Haar measure of G is given by $d(x, h) = |\det(h)|^{-1} dx dh$, and the modular function of G is given by $\Delta_G(x, h) = \Delta_H(h) |\det(h)|^{-1}$. G acts unitarily on $L^2(\mathbb{R}^d)$ by the **quasi-regular representation** defined by

$$[\pi(x, h)f](y) = |\det(h)|^{-1/2} f(h^{-1}(y - x)) . \quad (4)$$

We assume that H is chosen such that π is an **(irreducible) square-integrable representation**. Square-integrability of the representation means that there exists at least one nonzero

admissible vector $\psi \in L^2(\mathbb{R}^n)$ such that the matrix coefficient

$$(x, h) \mapsto \langle \psi, \pi(x, h)\psi \rangle$$

is in $L^2(G)$, which is the L^2 -space associated to a left Haar measure on G . In this case the associated wavelet transform

$$\mathcal{W}_\psi : L^2(\mathbb{R}^d) \ni f \mapsto ((x, h) \mapsto \langle f, \pi(x, h)\psi \rangle)$$

is a scalar multiple of an isometry, which gives rise to the **wavelet inversion formula**

$$f = \frac{1}{c_\psi} \int_G \mathcal{W}_\psi f(x, h) \pi(x, h) d\mu_G(x, h) . \quad (5)$$

A thorough understanding of the properties of the wavelet transform hinges on the *dual action*, i.e., the (right) linear action $\mathbb{R}^d \times H \ni (\xi, h) \mapsto h^T \xi$: By the results of [12, 15], H is admissible iff the dual action has a single open orbit $\mathcal{O} = \{h^T \xi_0 : h \in H\} \subset \mathbb{R}^d$ of full measure (for some $\xi_0 \in \mathcal{O}$), such that in addition the stabilizer group $H_{\xi_0} = \{h \in H : h^T \xi_0 = \xi_0\}$ is compact. (This condition does of course not depend on $\xi_0 \in \mathcal{O}$.) The dual orbit will be of central importance to this paper.

Let us next describe the pertinent notions from coorbit theory. A weight on a locally compact group K is a continuous function $w : K \rightarrow \mathbb{R}^+$ satisfying $w(xy) \leq w(x)w(y)$, for all $x, y \in K$. The Besov-type coorbit spaces that we focus on in this paper are obtained by fixing a weight v of the type

$$v(x, h) = (1 + |x| + \|h\|_\infty)^s w(h) \quad (6)$$

on G , where $|\cdot| : \mathbb{R}^d \rightarrow \mathbb{R}$ is an arbitrary fixed norm, and w is some weight on H . (Note that this indeed defines a weight v .) We then define, for $1 \leq p, q < \infty$,

$$L_v^{p,q}(G) = \left\{ F : G \rightarrow \mathbb{C} : \int_H \left(\int_{\mathbb{R}^d} |F(x, h)|^p v(x, h)^p dx \right)^{q/p} dh < \infty \right\} ,$$

with the obvious norm, and the usual conventions regarding identification of a.e. equal functions. We write $L_v^p(G) = L_v^{p,p}(G)$. The corresponding spaces for $p = \infty$ and/or $q = \infty$ are defined by replacing integrals with essential suprema. We will also use

$$L_s^p(\mathbb{R}^d) = \left\{ f \text{ Borel-measurable} : \int_{\mathbb{R}^d} |f(x)|^p (1 + |x|)^{sp} dx < \infty \right\} .$$

We next recall the necessary ingredients of coorbit theory. Our main sources for the following are the papers [7, 8, 9]. We assume that Y is a Banach space of functions on G that fulfills the conditions of [17, 2.2], i.e. it is continuously embedded in $L_{loc}^1(G)$, and fulfills certain compatibility conditions with convolution. Examples of such spaces are the $L_v^{p,q}(G)$ defined above. The following definition will be important:

Definition 2.1. *A weight v_0 is called **control weight** for Y if it satisfies*

$$v_0(x, h) = \Delta_G(x, h)^{-1} v_0((x, h)^{-1}) ,$$

as well as

$$\max \left(\|L_{(x,h)} \pm 1\|_{Y \rightarrow Y}, \|R_{(x,h)}\|_{Y \rightarrow Y}, \|R_{(x,h)}^{-1}\|_{Y \rightarrow Y} \Delta_G(x, h)^{-1} \right) \leq v_0(x, h)$$

where $L_{(x,h)}, R_{(x,h)}$ are left and right translation by $(x, h) \in G$.

Using a control weight v_0 for Y we define the set

$$\mathcal{A}_{v_0} = \{\psi \in L^2(\mathbb{R}^d) : \mathcal{W}_\psi \psi \in L^1_{v_0}(G)\}$$

of analyzing vectors. It turns out that \mathcal{A}_{v_0} is a vector space, in fact a Banach space, and invariant under π . We denote its conjugate dual as $\mathcal{A}_{v_0}^\sim$. The sesquilinear map $L^2(\mathbb{R}^d) \times \mathcal{A}_{v_0} \ni (f, g) \mapsto \langle f, g \rangle$ can be uniquely extended to $\mathcal{A}_{v_0}^\sim \times \mathcal{A}_{v_0}$. Hence, if we fix $\psi \in \mathcal{A}_{v_0}$, the definition of the continuous wavelet transform of $f \in \mathcal{A}_{v_0}^\sim$ via

$$\mathcal{W}_\psi f(x, h) = \langle f, \pi(x, h)\psi \rangle$$

again makes sense.

Now the coorbit space associated to Y is defined by fixing $\psi \in \mathcal{A}_{v_0}$ and letting

$$\text{Co}(Y) = \{f \in \mathcal{A}_{v_0}^\sim : \mathcal{W}_\psi f \in Y\}$$

with the norm $\|f\|_{\text{Co}(Y)} = \|\mathcal{W}_\psi f\|_Y$. It now follows by [8, Theorem 5.2] that the space $\text{Co}Y$ is a Banach space, and independent of the choice of the analyzing vector, as well as of the precise choice of control weight.

Clearly, the whole construction hinges on the actual existence of a nonzero analyzing vector. For arbitrary control weights v_0 , this might be difficult to answer. For the space $Y = L^{p,q}_v(G)$, with v as in (6), we first note that by [16, Lemma 2.3], a control weight for Y is given by

$$v_0(x, h) = (1 + |x| + \|h\|_\infty)^s w_0(h) .$$

where

$$w_0(h) = \max(\Delta_G(0, h)^{-1}, \Delta_G(0, h)) (1 + \|h^{-1}\|_\infty)^s ((w(h) + w(h)^{-1}) . \quad (7)$$

But then Theorem 2.1 of [16] implies that \mathcal{A}_{v_0} is nontrivial, and thus $\text{Co}Y$ is indeed well-defined. We next turn to the study of atomic decompositions. As already explained in the introduction, the sampling theorems derived in [7, 8, 9, 17] are rather flexible in terms of the sampling sets, at the price of imposing additional conditions on the analyzing vector. These are formulated in the next definition.

Definition 2.2. *Let Y denote any solid Banach function space on the locally compact group G , $U \subset G$ a compact neighborhood of the identity, and $F : G \rightarrow \mathbb{C}$. We let*

$$\mathcal{M}_U^R F(x) = \sup_{y \in U} |f(xy)|$$

denote the maximal function of F with respect to U . Given a weight v_0 on G , we denote the associated Wiener amalgam space by

$$W^R(C^0, L^1_{v_0}) = \{F : G \rightarrow \mathbb{C} : \mathcal{M}_U^R F \in L^1_{v_0}(G)\} ,$$

with norm $\|F\|_{W^R(C^0, L^1_{v_0})} = \|\mathcal{M}_U^R F\|_{L^1_{v_0}}$.

We let

$$\mathcal{B}_{v_0} = \{\psi \in L^2(\mathbb{R}^d) : \mathcal{W}_\psi \psi \in W^R(C^0, L^1_{v_0})\} .$$

Since $L_{v_0}^1(G) \supset W^R(C^0, L_{v_0}^1)$, every element in \mathcal{B}_{v_0} can be used to characterize elements of CoY , whenever v_0 is a control weight for Y . The additional condition will allow to derive the desired discrete characterizations as well. In other words, the elements of \mathcal{B}_{v_0} , for a control weight v_0 associated to a Banach function space Y , are precisely the “nice” wavelets associated to the coorbit space $Co(Y)$ that were mentioned in the introduction.

Definition 2.3. Let $U \subset G$ denote a neighborhood of the identity, and $Z = (z_i)_{i \in I} \subset G$.

- (a) The family $(z_i)_{i \in I}$ is called **U -dense**, if $\bigcup_{i \in I} z_i U = G$.
- (b) The family $(z_i)_{i \in I}$ is called **U -separated**, if $z_i U \cap z_j U = \emptyset$, whenever $i \neq j$. It is called **separated**, if there exists a neighborhood U of unity such that it is U -separated. It is called **relatively separated** if it is the finite union of separated families.

Remark 2.4. Note that U -dense, relatively separated families always exist, for every neighborhood U of the identity. More precisely, given any separated family Z_0 , there exists a U -dense separated set Z containing Z_0 . To see this, pick a neighborhood V of the identity such that Z_0 is V -separated, and in addition fulfills $V^2 = \{vw : v, w \in V\} \subset U$, and apply Zorn’s Lemma to find a V -separated family $Z = (z_i)_{i \in I}$ containing Z_0 that is maximal with respect to inclusion. Then maximality and $V^2 \subset U$ imply that this family is U -dense.

For the formulation of the atomic decomposition result, we need a norm on the sequences space. We use $\mathbf{1}_W : G \rightarrow \mathbb{R}$ to denote the indicator function of a subset $W \subset G$.

Definition 2.5. Let Y be a solid Banach function space on G , and $Z \subset G$ a relatively separated set. Picking an arbitrary compact neighborhood W of the identity, we define

$$\|(c_z)_{z \in Z}\|_{Y_d} = \left\| \sum_{z \in Z} c_z \mathbf{1}_{zW} \right\|_Y,$$

and let $Y_d = \{(c_z)_{z \in Z} \in \mathbb{C}^Z : \|(c_z)_{z \in Z}\|_{Y_d} < \infty\}$.

Remark 2.6. Note that Y_d is a nontrivial Banach space, and the norm of Y_d is (up to equivalence) independent of the choice of W [7]. For the semidirect product group $G = \mathbb{R}^d \rtimes H$, the typical sampling sets are of the type

$$Z = \{(h_j x_k, h_j) : j \in J, k \in K\}$$

where $\{h_j : j \in J\} \subset H$ and $\{x_k : k \in K\} \subset \mathbb{R}^d$ are uniformly dense and separated. One easily verifies that Z is uniformly dense and separated in G . Indeed, if $V \subset \mathbb{R}^d$ and $W \subset H$ separate $(x_k)_{k \in K}$ and $(h_j)_{j \in J}$, respectively, then $U = \{(x, g) : x \in V, g \in W\}$ separates Z : $(h_j x_k, h_j)U \cap (h_{j'} x_{k'}, h_{j'}) \neq \emptyset$ entails $h_j W \cap h_{j'} W \neq \emptyset$, and thus by assumption on W , $j = j'$. But then a comparison of the translation coordinates yields $h_j x_k + h_j V \cap h_j x_{k'} + h_j V \neq \emptyset$, i.e., $x_k + V \cap x_{k'} + V \neq \emptyset$, and thus $k = k'$. A similar calculation shows that Z is U -dense if $\{x_k : k \in K\}$ is V -dense and $\{h_j : j \in J\}$ is W -dense.

If $Y = L_v^{p,q}(G)$, for some weight v , the associated coefficient space norm is just (equivalent to) the discrete weighted $\ell^{p,q}$ -norm, i.e.

$$\|(c_{j,k})\|_{Y_d} \asymp \left(\sum_{j \in J} \left(\sum_{k \in K} |c_{j,k}|^p v(h_j x_k, h_j)^p \right)^{q/p} \right)^{1/q} \quad (8)$$

with the usual modifications for $p = \infty$ and/or $q = \infty$. To see this, note that we can pick a neighborhood $W = W_1 \times W_2 \subset \mathbb{R}^d \times H$ of the neutral element such that $(h_j x_k, h_j)W \cap (h_{j'} x_{k'}, h_{j'})W = \emptyset$. By submultiplicativity and continuity of the weight w , there are constants c_1, c_2 such that

$$\forall j \in J \forall k \in K : c_1 v(h_j x_k, h_j) \leq v|_{(h_j x_k, h_j)W} \leq c_2 v(h_j x_k, h_j) .$$

Hence if we employ this particular set W , left-invariance of Haar measure on G ensures (8), and we have $Y_d = \ell_v^{p,q}(Z)$. In particular, the finitely supported sequences are dense in Y_d .

We can now formulate the atomic decomposition result [17], which will be seen to imply Theorem 1.2. Note in particular that, since $L^2(\mathbb{R}^d) = \text{Co}L^2(G)$ by [8, Corollary 4.4], this statement will also yield criteria for wavelet frame generators of $L^2(\mathbb{R}^d)$.

Theorem 2.7. *Let v_0 denote a control weight for Y , and let $0 \neq \psi \in \mathcal{B}_{v_0}$. Then there exists a neighborhood $U \subset G$ of unity such that for all U -dense, relatively separated families $(z_i)_{i \in I} \subset G$, the following statements are true:*

(a) *There is a linear bounded map $C : \text{Co}Y \rightarrow Y_d(Z)$ with the property that, for all $f \in \text{Co}Y$,*

$$f = \sum_{i \in I} C(f)(z_i) \pi(z_i) \psi ,$$

with unconditional convergence in $\|\cdot\|_{\text{Co}Y}$.

(b) *Conversely, for every sequence $(c(z_i))_{i \in I} \in Y_d(Z)$, the sum*

$$g = \sum_{i \in I} c(z_i) \pi(z_i) \psi$$

converges unconditionally in $\|\cdot\|_{\text{Co}Y}$, with $\|g\|_{\text{Co}Y} \preceq \|(c(z_i))_{i \in I}\|_{Y_d(Z)}$.

(c) *The norms $\|f\|_{\text{Co}Y}$ and $\|(\mathcal{W}_\psi f(z_i))_{i \in I}\|_{Y_d(Z)}$ are equivalent. Moreover, $f \in \text{Co}Y$ iff $(\mathcal{W}_\psi f(z_i))_{i \in I} \in Y_d(Z)$.*

We next exhibit a class of nice wavelets: Any bandlimited Schwartz function with Fourier support contained in the open dual orbit is in \mathcal{B}_{v_0} . This was shown in [16], using oscillation estimates. The connection between oscillation and Wiener amalgam space is explained in the next remark:

Remark 2.8. *Let $U \subset G$ denote a relatively compact neighborhood of the identity, and $F : G \rightarrow \mathbb{C}$ any function. We let*

$$\text{osc}_U(F)(x) = \sup\{|F(x) - F(xy)| : y \in U\} .$$

It is easy to see that $F \in W^R(C^0, L^1_{v_0})$ holds iff both F and $\text{osc}_U(F)$ are in $L^1_{v_0}(G)$.

The following is [16, Lemma 2.6].

Theorem 2.9. *For any weight v_0 satisfying $v_0(x, h) \leq (1 + |x| + \|h\|_\infty)^s w_0(h)$, with $w_0 : H \rightarrow \mathbb{R}^+$ an arbitrary weight, we have $\mathcal{F}^{-1}(C_c^\infty(\mathcal{O})) \subset \mathcal{B}_{v_0}$.*

3 Formulation and proof of the main result

The results of the previous section have set the stage for the main result. While bandlimited Schwartz functions are fairly convenient to work with for many purposes, the usefulness of compactly supported wavelets has been emphasized repeatedly. The chief aim of this section is to replace the assumption of proper confinement of the supports on the Fourier transform by a more quantitative version in terms of decay properties of the Fourier transform $\widehat{\psi}(\xi)$, as ξ approaches the boundary of the dual orbit. These decay conditions will be formulated by vanishing moment conditions, defined as follows:

Definition 3.1. *Let $r \in \mathbb{N}$ be given. $f \in L^1(\mathbb{R}^d)$ has vanishing moments in \mathcal{O}^c of order r if all distributional derivatives $\partial^\alpha \widehat{f}$ with $|\alpha| < r$ are continuous functions, identically vanishing on \mathcal{O}^c .*

Note that under suitable integrability conditions on ψ , the vanishing moment conditions are equivalent to

$$\forall |j| < k, \forall \xi \in \mathcal{O}^c : \int_{\mathbb{R}^d} x^j \psi(x) e^{-2\pi i \langle \xi, x \rangle} dx = 0 .$$

We next define an auxiliary function $A : \mathcal{O} \rightarrow \mathbb{R}^+$ as follows: Given any point $\xi \in \mathcal{O}$, let $\text{dist}(\xi, \mathcal{O}^c)$ denote the minimal distance of ξ to \mathcal{O}^c , and define

$$A(\xi) = \min \left(\frac{\text{dist}(\xi, \mathcal{O}^c)}{1 + \sqrt{|\xi|^2 - \text{dist}(\xi, \mathcal{O}^c)^2}}, \frac{1}{1 + |\xi|} \right) .$$

By definition, A is a continuous function with $A(\cdot) \leq 1$.

Using A , we then define $\Phi_\ell : H \rightarrow \mathbb{R}^+ \cup \{\infty\}$ via

$$\Phi_\ell(h) = \int_{\mathbb{R}^d} A(\xi)^\ell A(h^T \xi)^\ell d\xi$$

We note a few simple but useful properties of Φ_ℓ :

Lemma 3.2. (a) $\ell \leq \ell'$ implies $\Phi_\ell(h) \geq \Phi_{\ell'}(h)$, for all $h \in H$.

(b) $\Phi_\ell(h) = |\det(h)|^{-1} \Phi_\ell(h^{-1})$.

Proof. Part (a) follows from $A(\cdot) \leq 1$, and part (b) by substitution: Letting $\omega = h^T \xi$, and using the notation $h^{-T} = (h^{-1})^T$, we find

$$\begin{aligned} \Phi_\ell(h) &= \int_{\mathbb{R}^d} A(\xi)^\ell A(h^T \xi)^\ell d\xi \\ &= |\det(h)|^{-1} \int_{\mathbb{R}^d} A(h^{-T} \omega) A(\omega) d\omega = |\det(h)|^{-1} \Phi_\ell(h^{-1}) . \end{aligned}$$

□

Now the following definition will allow to formulate sufficient vanishing moment criteria for elements of \mathcal{B}_{w_0} .

Definition 3.3. Let $w_0 : H \rightarrow \mathbb{R}^+$ denote a weight, $s \geq 0$. We call \mathcal{O} **strongly** (s, w_0) -temperately embedded (with index $\ell \in \mathbb{N}$) if $\Phi_\ell \in W(C^0, L^1_m)$, where the weight $m : H \rightarrow \mathbb{R}^+$ is defined by

$$m(h) = w_0(h) |\det(h)|^{-1/2} (1 + \|h\|_\infty)^{2(s+d+1)} .$$

This definition plays a similar role as the notion of *temperately embedded orbits* introduced in [16]. Essentially, the index ℓ will determine the number of vanishing moments needed to ensure that a given function is in \mathcal{B}_{w_0} . Note that somewhat contrary to the intuition conveyed by the terminology, it is currently not clear whether strong temperate embeddedness implies temperate embeddedness.

Theorem 3.4. Assume that \mathcal{O} is strongly (s, w_0) -embedded with index ℓ . Then any function $\psi \in L^1(\mathbb{R}^d) \cap C^{\ell+d+1}(\mathbb{R}^d)$ with vanishing moments in \mathcal{O}^c of order $t > \ell + s + d$ and integrable partial derivatives of order up to $2t$ is contained in \mathcal{B}_{w_0} .

There exist compactly supported functions ψ satisfying this condition.

Before we prove the theorem, we need various auxiliary results.

Lemma 3.5. Let m denote an arbitrary weight on a locally compact group G , f a continuous function on G , and U a compact neighborhood of the identity. Then there are constants c_1, c_2 , depending only on U and m , such that

$$c_1 \mathcal{M}_U^R(m \cdot f)(x) \leq m(x) \mathcal{M}_U^R(f)(x) \leq c_2 \mathcal{M}_U^R(m \cdot f)(x) .$$

Proof. Using submultiplicativity and continuity of m , one readily verifies the estimates with

$$c_1 = \frac{1}{\sup_{y \in U} m(y)} , \quad c_2 = \sup_{y \in U} m(y^{-1}) .$$

□

We next cite a useful result [16, Lemma 3.5], which explains the usefulness of the auxiliary functions: They serve as envelope functions for the Fourier transform $\widehat{\psi}, \widehat{f}$ and their derivatives, which can be translated to decay estimates for wavelet coefficients.

Lemma 3.6. Let α be a multiindex with $|\alpha| < r$. Assume that $f, \psi \in L^1(\mathbb{R}^d)$ have vanishing moments of order r in \mathcal{O}^c , and fulfill $|\widehat{f}|_{r, r-|\alpha|} < \infty, |\widehat{\psi}|_{r, r-|\alpha|} < \infty$. Then there exists a constant $C > 0$, independent of f and ψ , such that

$$|\partial^\alpha(\widehat{f} \cdot D_h \widehat{\psi})(\xi)| \leq C |\widehat{f}|_{r, r-|\alpha|} |\widehat{\psi}|_{r, r-|\alpha|} |\det(h)|^{1/2} (1 + \|h\|)^{|\alpha|} A(\xi)^{r-|\alpha|} A(h^T \xi)^{r-|\alpha|}$$

Now the following estimate reveals the usefulness of the auxiliary function Φ_ℓ , by giving an estimate of the wavelet coefficient $\mathcal{W}_\psi \psi$. The chief advantage of this estimate is that it separates the translation and dilation variables. Note that the same estimate applies also to $\mathcal{W} \psi_f$, as long as both ψ and f fulfill the conditions of the Lemma.

Lemma 3.7. Let $0 < m < r$, and let $\psi \in L^1(\mathbb{R}^d)$ denote a function with vanishing moments of order r in \mathcal{O}^c and $|\widehat{\psi}|_{r, r} < \infty$. Then

$$|\mathcal{W}_\psi \psi(x, h)| \prec |\widehat{\psi}|_{r, r}^2 (1 + |x|)^{-m} |\det(h)|^{1/2} (1 + \|h\|_\infty)^m \Phi_{r-m}(h) .$$

Proof. Since

$$\mathcal{W}_\psi \psi(x, h) = |\det(h)|^{1/2} \left(\widehat{\psi} \cdot (D_h \widehat{\psi}) \right)^\vee(x),$$

we can use the standard estimate of decay on the space side by L^1 -norms of derivatives on the Fourier transform side to obtain

$$\begin{aligned} |\mathcal{W}_\psi \psi(x, h)| &\preceq |\det(h)|^{1/2} (1 + |x|)^{-m} \sum_{|\alpha| \leq m} \left\| \partial^\alpha \left(\widehat{\psi} \cdot D_h \widehat{\psi} \right) \right\|_1 \\ &\preceq (1 + |x|)^{-m} |\det(h)|^{1/2} \sum_{|\alpha| \leq m} |\widehat{\psi}|_{r, r-|\alpha|}^2 (1 + \|h\|_\infty)^{|\alpha|} \underbrace{\int_{\mathbb{R}^d} A(\xi)^{r-|\alpha|} A(h^T \xi)^{r-|\alpha|} d\xi}_{=\Phi_{r-|\alpha|}(h)}, \end{aligned}$$

where we used Lemma 3.6. Now the monotonicity properties of the Schwartz norm and the Φ_ℓ (with respect to their subscripts) allow to estimate

$$\sum_{|\alpha| \leq m} |\widehat{\psi}|_{r, r-|\alpha|}^2 (1 + \|h\|_\infty)^{|\alpha|} \Phi_{r-|\alpha|}(h) \preceq |\widehat{\psi}|_{r, r}^2 (1 + \|h\|_\infty)^m \Phi_{r-m}(h),$$

which finishes the proof. \square

Proof of 3.4: We fix $V = B_1(x)$ and $W = \{h \in H : \|h - \text{id}\|_\infty < 1/2\}$. Then $U = V \times W \subset G$ is a neighborhood of the identity in G , and we will use this neighborhood to show finiteness of the amalgam norm $\|\mathcal{W}_\psi \psi\|_{W^R(C^0, L_{v_0}^1)}$. To this end, let $k = t - \ell > s + d$, and use Lemma 3.7 to estimate as follows:

$$\begin{aligned} &\|\mathcal{W}_\psi \psi\|_{W^R(C^0, L_{v_0}^1)} \\ &\leq \int_H \int_{\mathbb{R}^d} \sup_{y \in V, g \in W} |\mathcal{W}_\psi \psi(x + hy, hg)| (1 + |x|)^s dx \, w_0(h) \frac{dh}{|\det(h)|} \\ &\preceq \int_H \int_{\mathbb{R}^d} \sup_{y \in V, g \in W} \left((1 + |x + hy|)^{-k} (1 + |x|)^s \right. \\ &\quad \left. (1 + \|hg\|_\infty)^k |\det(hg)|^{1/2} \int_{\mathbb{R}^d} A(\xi)^{t-k} A((hg)^T \xi)^{t-k} d\xi \right) dx \, w_0(h) \frac{dh}{|\det(h)|} \\ &\leq \int_H \int_{\mathbb{R}^d} \left(\sup_{y \in V} (1 + |x + hy|)^{-k} \right) (1 + |x|)^s dx \left(\sup_{g \in W} \Psi(hg) \right) w_0(h) \frac{dh}{|\det(h)|}, \end{aligned} \quad (9)$$

where we introduced the auxiliary function

$$\Psi(h) = (1 + \|h\|_\infty)^k |\det(h)|^{1/2} \Phi_{t-k}(h).$$

Next, using that V is the unit ball, we find $|x + hy| \geq \max(|x| - |hy|, 0) \geq \max(|x| - \|h\|_\infty, 0)$, and thus

$$\sup_{y \in V} (1 + |x + hy|)^{-k} \leq (1 + \max(0, |x| - \|h\|_\infty))^{-k}.$$

Hence

$$\begin{aligned}
& \int_{\mathbb{R}^d} \left(\sup_{y \in V} (1 + |x + hy|)^{-k} (1 + |x|)^s \right) dx \\
& \leq \int_{\mathbb{R}^d} (1 + \max(0, |x| - \|h\|_\infty))^{-k} (1 + |x|)^s dx \\
& = \int_{|x| \leq \|h\|_\infty} (1 + |x|)^s dx + \int_{|x| > \|h\|_\infty} (1 + |x| - \|h\|_\infty)^{-k} (1 + |x|)^s dx \\
& \leq C_1 (1 + \|h\|_\infty)^s \|h\|_\infty^d + \int_{\mathbb{R}^d} (1 + |x| - \|h\|_\infty)^{-k} (1 + |x|)^s dx,
\end{aligned}$$

with C_1 denoting the volume of the unit ball. Now the estimate $(1 + |x| - \|h\|_\infty) (1 + \|h\|_\infty) \geq (1 + |x|)$ yields

$$\int_{\mathbb{R}^d} (1 + |x| - \|h\|_\infty)^{-k} (1 + |x|)^s dx \leq (1 + \|h\|_\infty)^k \underbrace{\int_{\mathbb{R}^d} (1 + |x|)^{s-k} dx}_{=: C_2}$$

with finite constant C_2 since $k - s > d$, and we finally arrive at

$$\int_{\mathbb{R}^d} \left(\sup_{y \in V} (1 + |x + hy|)^{-k} \right) (1 + |x|)^s dx \leq C (1 + \|h\|_\infty)^k. \quad (10)$$

We can now wrap up the proof: Plugging (10) into (9), and applying Lemma 3.5 to the maximal function $\sup_{g \in W} \Psi(hg) = \mathcal{M}_U^R \Psi(h)$, we obtain

$$\begin{aligned}
& \|\mathcal{W}_\psi \psi\|_{W^R(C^0, L_{v_0}^1)} \\
& \leq \int_H (1 + \|h\|_\infty)^k \mathcal{M}_U^R(\Psi)(h) w_0(h) \frac{dh}{|\det(h)|} \\
& \leq \int_H \mathcal{M}_U^R(\Phi_{t-k})(h) (1 + \|h\|_\infty)^{2k} w_0(h) |\det(h)|^{1/2} dh \\
& = \|\Phi_{t-k}\|_{W^R(C^0, L_m^1)}.
\end{aligned}$$

Since $t - k \geq \ell$, the final expression is finite by assumption, and the sufficient criterion for $\psi \in \mathcal{B}_{v_0}$ is established.

Regarding existence of compactly supported atoms, we recall from [16, Lemma 4.1] the existence of a partial differential operator D of order $k \leq 2d$ with constant coefficients and the property that, for all $\rho \in L^1(G)$ with integrable partial derivatives of sufficiently high order, $D^t \rho$ has vanishing moments of order t . Thus picking $\rho \in C_c^\infty(\mathbb{R}^d)$ and letting $\psi = D^t \rho$ yields the desired compactly supported atom. \square

4 Verifying strong temperate embeddedness

Theorem 3.4 shows that explicit vanishing moment conditions for elements of \mathcal{B}_{v_0} , given a concrete dilation group H and control weight v_0 , can be obtained by the following steps:

1. Compute the dual orbit \mathcal{O} , and the auxiliary function $A : \mathcal{O} \rightarrow \mathbb{R}^+$.

2. Compute an upper estimate of

$$\Phi_\ell : H \ni h \mapsto \int_{\mathbb{R}^d} A(\xi) A(h^T \xi) d\xi ,$$

i.e., determine $\tilde{\Phi}_\ell : H \rightarrow \mathbb{R}^+$ with $\Phi_\ell \preceq \tilde{\Phi}_\ell$.

3. Determine Ψ_ℓ with $\mathcal{M}_U^R \tilde{\Phi}_\ell(\xi) \preceq \Psi_\ell(\xi)$.

4. Determine an explicit ℓ such that

$$\int_H \Psi_\ell(h) w_0(h) |\det(h)|^{-1/2} (1 + \|h\|_\infty)^{2(s+d+1)} dh < \infty .$$

In this section we will see that this program can indeed be carried out for a large class of dilation groups, leading to concrete criteria. In particular, our results will cover all admissible dilation groups in dimension two. Note that there is some freedom of choice in picking the upper bounds $\tilde{\Phi}_\ell$ and Ψ_ℓ .

Remark 4.1. *In the following calculations, we have typically tried to cover large classes of weights with minimal computational effort, possibly at the cost of suboptimal estimates for ℓ . Given any concrete group dilation H and a control weight v_0 on G of particular interest, better estimates for ℓ may be possibly achieved by other methods.*

It should be mentioned at this point that the approach via Theorem 3.4 is probably not well-suited for obtaining optimal estimates for the number of vanishing moments. Note that the theorem is based on the decay estimate in Lemma 3.7 which uses the global behaviour of the wavelet and its Fourier transform. It seems likely that for the estimation of local quantities, such as $\mathcal{M}_U^R(\mathcal{W}_\psi \psi)$, this is not the sharpest available method. As an interesting alternative, we mention the techniques of [10], which achieve similar results with significantly less vanishing moments, but only for isotropic dilations.

4.1 The similitude groups

Let $H = \mathbb{R}^+ \cdot SO(d) \subset GL(d, \mathbb{R})$ denote the similitude group. This was the first higher-dimensional dilation group used for the construction of continuous wavelet transforms [1, 21]. The open dual orbit is easily computed as $\mathcal{O} = \mathbb{R}^d \setminus \{0\}$, hence we find $\mathcal{O}^c = \{0\}$. This implies

$$A(\xi) = \min \left(|\xi|, \frac{1}{1 + |\xi|} \right) .$$

We write elements $h \in H$ as $h = rS$, with $r > 0$ and $S \in SO(d)$. Haar measure on H is then given by $dh = \frac{dr}{r} dS$; here dS denotes integration against Haar measure on $SO(d)$ normalized to one. The following lemma provides the central estimate of the auxiliary function Φ_ℓ :

Lemma 4.2. *For $h = rS \in H$ and $\ell > d$, we have*

$$\Phi_\ell(h) \leq C \min(r^{\ell-d}, r^{d-\ell}) .$$

Proof. First note that by invariance of the euclidean distance under elements of $SO(d)$, we have $A(h\xi) = A(r\xi)$, and thus $\Phi_\ell(h) = \Phi_\ell(r \cdot \text{id}_{\mathbb{R}^d})$, and integration in polar coordinates yields

$$\begin{aligned}\Phi_\ell(h) &= \int_{\mathbb{R}^d} A(\xi)^\ell A(h^T \xi)^\ell d\xi \\ &= C_1 \int_0^\infty \min\left(s, \frac{1}{1+s}\right)^\ell \min\left(rs, \frac{1}{1+rs}\right)^\ell s^{d-1} ds ,\end{aligned}$$

where C_1 is the surface of the unit ball. Let us now assume that $r \leq 1$; the case $r > 1$ will then be addressed using Lemma 3.2(b). First note that

$$\min\left(s, \frac{1}{1+s}\right) = \begin{cases} s & s < c \\ \frac{1}{1+s} & s > c \end{cases}$$

where we used $c = \frac{\sqrt{3}-1}{2}$. This implies

$$\min\left(s, \frac{1}{1+s}\right)^\ell \min\left(rs, \frac{1}{1+rs}\right)^\ell = \begin{cases} s^{2\ell+d+1} r^\ell & 0 < s < c \\ \frac{r^\ell s^{\ell+d+1}}{(1+s)^\ell} & c < s < c/r \\ \frac{s^{d-1}}{(1+s)^\ell (1+rs)^\ell} & c/r < s \end{cases} , \quad (11)$$

and hence

$$\Phi_\ell(r)/C_1 = \left(r^\ell \frac{s^{2\ell+d}}{2\ell+d} \right) \Big|_{s=0}^{s=c} + \underbrace{r^\ell \int_c^{c/r} \frac{s^{\ell+d-1}}{(1+s)^\ell} ds}_{=: I_1} + \underbrace{\int_{c/r}^\infty \frac{s^{d-1}}{(1+s)^\ell (1+rs)^\ell} ds}_{=: I_2} .$$

Now $\frac{s^\ell}{(1+s)^\ell} \leq 1$ implies

$$I_1 \leq r^\ell \int_c^{c/r} s^{d-1} ds = C_2 r^{\ell-d}$$

with a suitable constant $C_2 > 0$. Furthermore, for $s > c/r$, we have

$$\frac{s^{d-1}}{(1+s)^\ell} \leq (1+s)^{d-1-\ell} \leq (1+c/r)^{d-1-\ell} \leq C_3 r^{\ell+1-d}$$

which implies that

$$I_2 \leq \int_{c/r}^\infty \frac{C_3 r^{\ell+1-d}}{(1+rs)^\ell} ds = C_3 r^{\ell-d} \int_c^\infty \frac{1}{(1+s)^\ell} ds = C_4 r^{\ell-d} .$$

In summary, this yields

$$\Phi_\ell(h) \leq C_5 r^{\ell-d} .$$

If $r > 1$, then

$$\Phi_\ell(h) = |\det(h)|^{-1} \Phi_\ell(h^{-1}) \leq r^{-d} C_5 r^{d-\ell} = C_5 r^{-\ell} .$$

This finishes the proof. \square

Theorem 4.3. *Assume that the control weight on H fulfills*

$$w_0(h) \leq (r + r^{-1})^\beta$$

for some $\beta > 0$. Then the dual orbit is (s, w_0) -strongly temperately embedded, with index $\ell = \beta + 2s + \frac{5}{2}d + 3$.

Proof. Since the mapping $\Psi_\ell : h \mapsto \min(r^{d-\ell}, r^{\ell-d})$ is submultiplicative, Lemma 3.5 implies that $c_1 \Psi_\ell \leq \mathcal{M}_U^R \Psi_\ell \leq c_2 \Psi_\ell$, with suitable constants $0 < c_1 \leq c_2$. We have $|\det(h)| = r^d$ and $\|h\|_\infty = r$, and by the previous lemma $\Phi_\ell \leq \Psi_\ell$. Hence it is sufficient to prove that the integral

$$\int_0^\infty (r + r^{-1})^\beta (1 + r)^{2(s+d+1)} r^{-d/2} \min(r^{d-\ell}, r^{\ell-d}) \frac{dr}{r}$$

is finite. This is the case as soon as $\ell > \beta + 2s + \frac{5}{2}d + 2$. \square

We will next exhibit homogeneous Besov spaces as coorbit spaces over the similitude group, a fact that has already been noted in [7, 7.2]. Since the argument for higher dimensions is only sketched in [7], I include a short proof that combines results from [16] with the φ -transform characterization due to Frazier and Jawerth.

Theorem 4.4. *For all $1 \leq p, q \leq \infty$, we have $\dot{B}_{p,q}^\alpha(\mathbb{R}^d) = Co(L_v^{p,q}(G))$, with weight function*

$$v(x, h) = v(x, rS) = r^{-\alpha-d/2+d/p}.$$

Proof. We will use the φ -transform characterization of Frazier and Jawerth, see [11]. The φ -transform is based on the choice of two isotropic Schwartz functions φ, ψ with Fourier transforms compactly supported away from zero, and satisfying (amongst other properties)

$$\forall f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P} : f = \sum_{j \in \mathbb{Z}} f * \varphi_j * \psi_j.$$

Here $\varphi_j(x) = 2^{-j} \varphi(2^{-j}x) = 2^{-j/2}(\pi(0, 2^j)\varphi)(x)$, and ψ_j is defined analogously. $\mathcal{P} \subset \mathcal{S}'(\mathbb{R}^d)$ denotes the subspace of polynomials, and the series converges in $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}$. The associated discrete wavelet systems are then defined (in the terminology of this paper) via

$$\varphi_{j,k} = \pi(2^j k, 2^j) \varphi, \quad \psi_{j,k} = \pi(2^j k, 2^j) \psi.$$

Here we have already somewhat adapted the notation of [11] to the terminology of this paper, in particular the indexing conventions for small vs. large scales used here differ from [11]. The norms of the associated discrete coefficient spaces are defined by

$$\|(c_{j,k})_{j,k}\|_{b_{p,q}^\alpha} = \left(\sum_{j \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}^d} \left(2^{-j\alpha-jd/2+jd/p} |c_{j,k}| \right)^p \right)^{p/q} \right)^{1/q}$$

with the usual adjustments in the cases $p = \infty$ and/or $q = \infty$. We now define the separated subset

$$Z = \{(2^j k, 2^j \cdot \text{id}_{\mathbb{R}^d}) : j \in \mathbb{Z}, k \in \mathbb{Z}^d\} \subset G$$

which is in obvious bijective correspondence to $\mathbb{Z} \times \mathbb{Z}^d$. This bijection then induces an isometric isomorphism

$$\dot{b}_{p,q}^\alpha \cong \ell_v^{p,q}(Z)$$

which we will use to identify the two spaces at our convenience.

Now [11, Theorem 6.16] states that $f \in \dot{B}_{p,q}^\alpha(\mathbb{R}^d)$ iff $(\langle f, \varphi_{j,k} \rangle)_{j,k} \in \dot{b}_{p,q}^\alpha$, and we will use this characterization to show that $\dot{B}_{p,q}^\alpha(\mathbb{R}^d) = Co(L_v^{p,q}(G))$

To see the inclusion $\dot{B}_{p,q}^\alpha(\mathbb{R}^d) \subset Co(L_v^{p,q}(G))$, note that by 2.9, both φ and ψ are in the space \mathcal{B}_{v_0} , where v_0 denotes the control weight. Moreover, the set $\{(2^j k, 2^j) : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is a separated subset of G . Hence, if $f \in \dot{B}_{p,q}^\alpha(\mathbb{R}^d)$, the sum

$$\sum_{j,k} \langle f, \varphi_{j,k} \rangle \pi(2^j k, 2^j) \psi$$

converges in $Co(L_v^{p,q}(G))$, with respect to the norm of that space, by [8, Theorem 6.1(ii)]. Here we used $(\langle f, \varphi_{j,k} \rangle)_{j,k} \in \dot{b}_{p,q}^\alpha$, the observation that $Y_d = \ell_v^{p,q}(Z)$, and the density of the finitely supported sequences in that space (see Remark 2.6). Since $Co(L_v^{p,q}(G)) \subset \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}$ continuously (see the remarks following [16, Corollary 4.5]), we find that the $Co(L_v^{p,q})$ -limit of the sum coincides with f . Hence $f \in Co(L_v^{p,q}(G))$.

Conversely, assume that $f \in Co(L_v^{p,q}(G))$, then the fact that $\varphi \in \mathcal{B}_{v_0}$ allows to invoke Theorem 2.7. Hence, for sufficiently large $m \in \mathbb{N}$, and a suitable finite set $\mathcal{S} \subset SO(d)$ of rotations (which we can assume to contain the identity matrix), the set

$$Z' = \left\{ \left(2^{j/m} S \frac{k}{m}, 2^{j/m} S \right) : j, k \in \mathbb{Z}, S \in \mathcal{S} \right\} \subset G$$

will be such that 2.7(c) applies, yielding that

$$(\langle f, \pi(z') \varphi \rangle)_{z' \in Z'} \in \ell_v^{p,q}(Z').$$

But since $Z \subset Z'$, this implies $(\langle f, \varphi_{j,k} \rangle)_{j,k} \in \dot{b}_{p,q}^\alpha$, and thus $f \in \dot{B}_{p,q}^\alpha(\mathbb{R}^d)$. \square

Remark 4.5. *Let us now work out concrete vanishing moment conditions for atoms in $B_{p,q}^\alpha(\mathbb{R}^d)$. By [16, Lemma 2.3], a control weight for $L_v^{p,q}(G)$ is given by*

$$\begin{aligned} w_0(x, h) &= \max(\Delta_G(0, h)^{-1}, \Delta_G(0, h)) (r^{-\alpha-d/2+d/p} + r^{\alpha+d/2-d/p}) \\ &= \max(r^{-d}, r^d) (r^{-\alpha-d/2+d/p} + r^{\alpha+d/2-d/p}) \\ &\leq (r + r^{-1})^{d+|\alpha-d/2+d/p|}. \end{aligned}$$

Thus vanishing moments of order $t > |\alpha - d/2 + d/p| + \frac{9}{2}d + 3$ will suffice, by Theorems 4.3 and 3.4. Applying this to the case $\alpha = d/2 - d/p$ and $p = q$, we obtain Theorem 1.2 by combining Theorem 2.7 with Proposition 1.1.

4.2 The diagonal groups

The diagonal group of dimension d is

$$H = \left\{ \left(\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_d \end{pmatrix} : \prod_{i=1}^d a_i \neq 0 \right) \right\}. \quad (12)$$

The open dual orbit is given by $\mathcal{O} = \{\xi = (\xi_1, \dots, \xi_d)^T \in \mathbb{R}^d : \prod_i \xi_i \neq 0\}$. The auxiliary function A is given by

$$A(\xi) = \min \left(\frac{\min_i |\xi_i|}{1 + \sqrt{|\xi|^2 - \min_i |\xi_i|^2}}, \frac{1}{1 + |\xi|} \right).$$

The control weights we are interested in are of the type

$$v_0(x, h) = (1 + |x| + \|h\|_\infty)^s w_0(h),$$

where $w_0(h) = \prod_{i=1}^d (a_i + a_i^{-1})^\alpha$. The diagonal group is in fact a special case of a diagonally acting direct product group. Hence we first prove the following, somewhat more general result, which is of independent interest.

Lemma 4.6. *Let $H_j < \text{GL}(d_j, \mathbb{R})$ be admissible dilation groups, with $j = 1, 2, \dots, k$, and $\sum_{j=1}^k d_j = d$, and let $H < \text{GL}(d, \mathbb{R})$ be defined as*

$$H = \left\{ \begin{pmatrix} h_1 & & 0 \\ & h_2 & \\ & & \ddots \\ 0 & & & h_k \end{pmatrix} : h_j \in H_j, j = 1, \dots, k \right\}.$$

Then H is admissible. Let v_0 denote a weight on $G = \mathbb{R}^d \rtimes H$, given by

$$v_0(x, h) = (1 + |x| + \|h\|_\infty)^s \prod_{j=1}^k w_j(h_j), \quad h = \begin{pmatrix} h_1 & & 0 \\ & h_2 & \\ & & \ddots \\ 0 & & & h_k \end{pmatrix},$$

with weights w_j on H_j . Let \mathcal{O}_j denote the open dual orbit of H_j , and \mathcal{O} the open dual orbit of H . If \mathcal{O}_j is (s, w_j) -temperately embedded with index ℓ_j , for $j = 1, 2, \dots, k$, then \mathcal{O} is (s, w) -temperately embedded with index $\ell = \max(\ell_1, \ell_2, \dots, \ell_k)^k$.

Proof. Obviously, H has a unique open orbit given by $\mathcal{O} = \prod_{j=1}^k \mathcal{O}_j$, and the associated fixed groups are the direct products of the fixed groups in H_j , respectively. These are compact by assumption, hence H is admissible as well.

Let $A : \mathcal{O} \rightarrow \mathbb{R}^+$ and $A_j : \mathcal{O}_j \rightarrow \mathbb{R}^+$ ($j = 1, \dots, k$) denote the auxiliary functions. Then we have, for $\xi = (\xi_1, \xi_2, \dots, \xi_k)$ with $\xi_j \in \mathcal{O}_j$:

$$A(\xi)^k \leq \prod_{j=1}^k A_j(\xi_j). \quad (13)$$

To see this, let $\xi'_j \in \mathcal{O}_j^c \subset \mathbb{R}^{d_j}$ denote elements of minimal distance to ξ_j , i.e. $|\xi_j - \xi'_j| = \text{dist}(\xi_j, \mathcal{O}_j^c)$.

Let $j_0 \in \{1, \dots, k\}$ be such that $|\xi_{j_0} - \xi'_{j_0}| = \min_j |\xi_j - \xi'_j|$, and let $\eta_j = \xi_j$ for $j \neq j_0$, as well as $\eta_{j_0} = \xi'_{j_0}$. Then $\xi' = (\eta_1, \dots, \eta_k)^T$ is an element of \mathcal{O}^c with minimal distance to ξ . We therefore obtain

$$A(\xi) = \min \left(\frac{|\xi_{j_0} - \xi'_{j_0}|}{1 + \sqrt{|\xi_{j_0}|^2 + \sum_{j \neq j_0} |\xi_j|^2}}, \frac{1}{1 + |\xi|} \right).$$

By [16, Lemma 3.1], $\mathbb{R}^+ \cdot \xi'_j \subset \mathcal{O}_j^c$, hence ξ_j and $\xi_j - \xi'_j$ are orthogonal, and we find in particular that $|\xi'_j| \leq |\xi_j|$, which entails

$$\frac{|\xi_{j_0} - \xi'_{j_0}|}{1 + \sqrt{|\xi'_{j_0}|^2 + \sum_{j \neq j'} |\xi_j|^2}} \leq \frac{|\xi_j - \xi'_j|}{1 + |\xi'_j|},$$

and since in addition $(1 + |\xi|) \geq (1 + |\xi_j|)$, this finally yields $A(\xi) \leq A_j(\xi_j)$. Taking the product over $j = 1, \dots, k$ then gives (13).

Now the desired result follows from (13) and Fubini's theorem, observing that left Haar measure on H is the product of left Haar measures on the H_j , together with $(1 + \|h\|_\infty)^s \leq \prod_{j=1}^k (1 + \|h_j\|_\infty)^s$. \square

We can now combine the lemma with Theorem 4.3, to obtain the following result.

Corollary 4.7. *Let H be defined by (12), and assume that the control weight on H fulfills*

$$w_0(h) \leq \prod_{a=1}^d (|a_j| + |a_j|^{-1})^\alpha$$

for some $\alpha > 0$. Then the dual orbit is (s, w_0) -strongly temperately embedded, with index $\ell = (\alpha + 2s + 11/2)^d$.

4.3 Shearlet groups in dimension two

Fix a real parameter c , and let

$$H = H_c = \left\{ \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}.$$

Here we use the convention $a^c = \text{sign}(a)|a|^c$ for $a < 0$. For $c = 1/2$, H_c is the shearlet group introduced in [19], and further studied (e.g.) in [3, 4], see also [20] for a comprehensive overview. The other groups are obviously closely related; the parameter c can be understood as controlling the anisotropy used in the scaling. Haar measure on H is given by $db \frac{da}{|a|^{2+c}}$, the modular function is $\Delta_H(h) = |a|^{2c}$. The dual orbit is computed as

$$\mathcal{O} = \mathbb{R}^2 \setminus (\{0\} \times \mathbb{R}).$$

For $h = \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \in H$ and $\xi_0 = (1, 0)^T \in \mathcal{O}$, we obtain $h^T \xi_0 = (a, b)^T$.

One computes

$$A(\xi) = \min \left(\frac{|\xi_1|}{1 + |\xi_2|}, \frac{1}{1 + |\xi|} \right).$$

We now come to the critical Step 2. in the program outlined at the beginning of this section. The following estimate is central to this subsection:

Lemma 4.8. *Let $\Phi_\ell : H \rightarrow \mathbb{R}^+$ be defined by*

$$\Phi_\ell(h) = \int_{\mathbb{R}^2} A(\xi)^\ell A(h^T \xi)^\ell d\xi.$$

Assume that $t, r_1, r_2 \geq 1$ are integers satisfying the inequality

$$t \geq 3r_1 + (3 + 6|c|)r_2 + 3|c| + 2. \quad (14)$$

Then, for all $h = \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \in H$, we have the estimate

$$\Phi_t(h) \leq C(|a| + |a|^{-1})^{-r_1} (1 + |b|)^{-r_2}, \quad (15)$$

with a suitable constant $C > 0$.

Proof. Let

$$u_1 = r_1 + (1 + 2|c|)r_2 + |c|, \quad u_2 = 2u_1 + 2 = 2r_1 + (2 + 3|c|)r_2 + 2|c| + 2. \quad (16)$$

Then we have $u_1 + u_2 \leq t$, and thus we obtain, for all $0 \leq u_3 \leq u_1$

$$\forall \xi = (\xi_1, \xi_2)^T \in \mathcal{O} : A(\xi)^t \leq \frac{|\xi_1|^{u_1}}{(1 + |\xi_1|)^{u_2}} \frac{1}{(1 + |\xi_2|)^{u_3}}. \quad (17)$$

For $\xi \in \mathcal{O}$, we have $h^T(\xi) = (a\xi_1, b\xi_1 + a^c\xi_2)^T$, and hence plugging (17) into the definition of Φ_t yields

$$\begin{aligned} \Phi_t(h) &= \int_{\mathbb{R}^d} A(\xi)^t A(h^T \xi)^t d\xi \\ &\leq \int_{\mathbb{R}} \frac{|\xi_1|^{u_1}}{(1 + |\xi_1|)^{u_2}} \frac{|a\xi_1|^{u_1}}{(1 + |a\xi_1|)^{u_2}} \\ &\quad \int_{\mathbb{R}} (1 + |\xi_2|)^{-u_3} (1 + |b\xi_1 + a^c\xi_2|)^{-u_3} d\xi_2 d\xi_1. \end{aligned} \quad (18)$$

We now employ the estimate

$$\int_{\mathbb{R}} (1 + |y|)^{-r} (1 + \alpha|x - y|)^{-r} dy \leq C (\alpha^{-1}(1 + |x|)^{-r} + (1 + \alpha|x|)^{-r}), \quad (19)$$

(see [5, Lemma 3.1]) with $r = u_3$, $\alpha = |a|^c$ and $x = a^{-c}b\xi_1$, to continue from (18) to obtain

$$\Phi_t(h) \leq C(I_1 + I_2)$$

with

$$I_1 = \int_{\mathbb{R}} \frac{|\xi_1|^{u_1}}{(1 + |\xi_1|)^{u_2}} \frac{|a\xi_1|^{u_1}}{(1 + |a\xi_1|)^{u_2}} \frac{|a|^{-c}}{(1 + |a^{-c}b\xi_1|)^{u_3}} d\xi_1$$

and

$$I_2 = \int_{\mathbb{R}} \frac{|\xi_1|^{u_1}}{(1 + |\xi_1|)^{u_2}} \frac{|a\xi_1|^{u_1}}{(1 + |a\xi_1|)^{u_2}} \frac{1}{(1 + |b\xi_1|)^{u_3}} d\xi_1.$$

Now

$$\begin{aligned} I_1 &= |a|^{u_1-c} \int_{\mathbb{R}} \frac{|\xi_1|^{2u_1}}{(1 + |\xi_1|)^{u_2} (1 + |a\xi_1|)^{u_2} (1 + |a^{-c}b\xi_1|)^{u_3}} d\xi_1 \\ &= |a|^{u_1-c} |a^{-c}b|^{-u_3} \int_{\mathbb{R}} \frac{|\xi_1|^{2u_1}}{(1 + |\xi_1|)^{u_2} (1 + |a\xi_1|)^{u_2}} \underbrace{\frac{|a^{-c}b\xi_1|^{u_3}}{(1 + |a^{-c}b\xi_1|)^{u_3}}}_{\leq 1} d\xi_1 \\ &\leq |a|^{u_1-c+cu_3} |b|^{-u_3} C \int_{\mathbb{R}} \frac{|\xi_1|^{2u_1}}{(1 + |\xi_1|)^{u_2}} d\xi_1. \end{aligned}$$

Our choice of constants implies $u_2 - 2u_1 \geq 2$, hence the integral converges. Since this holds for all $u_3 \leq r_2$, we thus obtain

$$I_1 \preceq |a|^{u_1 - |c|(1+r_2)} (1 + |b|)^{-r_2} . \quad (20)$$

For I_2 , we obtain with a similar calculation

$$\begin{aligned} I_2 &= |a|^{u_1} |b|^{-u_3} \int_{\mathbb{R}} \frac{|\xi_1|^{2u_1 - u_3}}{(1 + |\xi_1|)^{u_2} (1 + |a\xi_1|)^{u_2}} \underbrace{\frac{|b\xi_1|^{u_3}}{(1 + |b\xi_1|)^{u_3}}}_{\leq 1} d\xi_1 \\ &\preceq |a|^{u_1} |b|^{-u_3} , \end{aligned}$$

and since $0 \leq u_3 \leq r_2$ was arbitrary, we find

$$I_2 \preceq |a|^{u_1} (1 + |b|)^{-r_2} .$$

But this means that for the case $|a| \leq 1$ we have in fact established

$$\Phi_t(h) \preceq |a|^{u_1 - |c|(1+r_2)} (1 + |b|)^{-r_2} , \quad (21)$$

which is stronger than (15), since $u_1 \geq r_1 + |c|(1 + r_2)$.

In order to apply Lemma 3.2 to the case $|a| > 1$, we first compute $h^{-1} = \begin{pmatrix} a^{-1} & -a^{-c-1}b \\ 0 & a^{-c} \end{pmatrix}$.

Hence Lemma 3.2 and (21) yield

$$\begin{aligned} \Phi_t(h) &= \underbrace{|\det(h)|^{-1}}_{=|a|^{-1-c}} \Phi_t(h^{-1}) \\ &\preceq |a|^{-1-c} |a|^{-u_1 + |c|(1+r_2)} (1 + |a^{-c-1}b|)^{-r_2} \\ &\leq |a|^{-u_1 + |c|(1+r_2)} |a|^{(1+|c|)r_2} (1 + |b|)^{-r_2} \\ &= |a|^{-u_1 + |c|(1+r_2) + (1+|c|)r_2} (1 + |b|)^{-r_2} \\ &= |a|^{-r_1} (1 + |b|)^{-r_2} , \end{aligned}$$

by choice of u_1 . This proves the Lemma. \square

Remark 4.9. Combining Lemma 3.7 with (15) yields a decay estimate for shearlet coefficients, that is possibly of independent interest:

$$|\mathcal{W}_\psi f(x, h)| \leq C(1 + |x|)^{-m} (|a| + |a|^{-1})^{-r_1} (1 + |b|)^{-r_2} ,$$

which holds for all shearlets ψ and signals f fulfilling sufficient vanishing moment, smoothness and decay conditions. Note that the additional factors $(1 + \|h\|_\infty)^m$ and $|\det(h)|^{1/2}$ occurring in Lemma 3.7 are absorbed by suitably high powers of $(|a| + |a|^{-1})^{-1} (1 + |b|)^{-1}$.

The following theorem establishes strong temperate embeddedness. We use the same w_0 as in [3]; note that the parametrization of the dilation group used in that paper differs from the one employed here.

Theorem 4.10. Let $w_0(h) = (|a| + |a|^{-1})^{u_1} (|a| + |a|^{-1} + |a^c b|)^{u_2}$, for some $u > 0$. Then \mathcal{O} is strongly (s, w_0) -temperately embedded with index

$$\ell = \left\lceil 3u_1 + (9 + 12|c|)u_2 + 18(1 + |c|)s + 51|c| + \frac{151}{2} \right\rceil . \quad (22)$$

Proof. Introduce

$$r_2 = u_2 + 2s + 8$$

and

$$\begin{aligned} r_1 &= 2 + r_2 + u_1 + u_2(1 + |c|) + 2(1 + |c|)s + \frac{15}{2}|c| + \frac{17}{2} \\ &= u_1 + u_2(2 + |c|) + 2(2 + |c|)s + \frac{15}{2}|c| + \frac{33}{2} . \end{aligned}$$

Then one readily verifies

$$\ell \geq 3r_1 + (3 + 6|c|)r_2 + 3|c| + 2 ,$$

hence Lemma 4.8 is applicable.

We fix the neighborhood U of the identity element in H as

$$U = \left\{ \begin{pmatrix} a_1 & b_1 \\ 0 & a_1^c \end{pmatrix} : 1/2 < a_1 < 2 , |b_1| < 1 \right\} .$$

Given $h = \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \in H$, we employ (14) to estimate

$$\begin{aligned} \mathcal{M}_U^R(\Phi_t)(h) &\leq \sup \left\{ (|aa_1| + |aa_1|^{-1})^{-r_1} (1 + |ab_1 + a_1^c b|)^{-r_2} : 1/2 < a_1 < 2, |b_1| < 1 \right\} \\ &\leq 2^{r_1} (|a| + |a|^{-1})^{-r_1} \sup_{1/2 < a_1 < 2, |b| \leq 1} (1 + |ab_1 + a_1^c b|)^{-r_2} \\ &\leq (|a| + |a|^{-1})^{-r_1} (1 + |a|)^{r_2} (1 + |b|)^{-r_2} \\ &\leq (|a| + |a|^{-1})^{r_2 - r_1} (1 + |b|)^{-r_2} . \end{aligned}$$

A left Haar measure on H is given by $db \frac{da}{|a|^{2+c}}$, and instead of the operator norm, we take the equivalent norm $\|h\| = |a| + |a^c| + |b|$. Hence we need to estimate

$$\begin{aligned} I &= \int_{\mathbb{R}'} \int_{\mathbb{R}} (|a| + |a|^{-1})^{u_1} (|a| + |a|^{-1} + |a^c b|)^{u_2} |a|^{-(1+c)/2} \\ &\quad (1 + |a| + |a|^c + |b|)^{2(s+3)} (|a| + |a|^{-1})^{r_2 - r_1} (1 + |b|)^{-r_2} db \frac{da}{|a|^{2+c}} . \end{aligned}$$

Using the estimates

$$(|a| + |a|^{-1} + |a^c b|) \leq (|a| + |a|^{-1})^{1+|c|} (1 + |b|)$$

and

$$(1 + |a| + |a|^c + |b|) \leq (|a| + |a|^{-1})^{1+|c|} (1 + |b|) ,$$

we find that

$$I \leq \int_{\mathbb{R}} (|a| + |a|^{-1})^{e_1} da \int_{\mathbb{R}} (1 + |b|)^{e_2} db ,$$

with exponents

$$\begin{aligned} e_1 &= r_2 - r_1 + u_1 + u_2(1 + |c|) + 2(s + 3)(1 + |c|) + \frac{3}{2}|c| + \frac{5}{2} \\ &= r_2 - r_1 + u_1 + u_2(1 + |c|) + 2(1 + |c|)s + \frac{15}{2}|c| + \frac{17}{2} \end{aligned}$$

and

$$e_2 = -r_2 + u_2 + 2s + 6 .$$

Our choice of r_1, r_2 implies that $e_1, e_2 = -2$. Hence $I < \infty$, and we are done. \square

Remark 4.11. *In order to formulate an analog of Theorem 1.2, we want to explicitly determine a sufficient number of vanishing moments for atoms in $\text{Co}(\text{L}^p(G))$. For this purpose, first observe that for $h = \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix}$, the modular function of G given by $\Delta_G(x, h) = \frac{\Delta_H(h)}{|\det(h)|} = |a|^{c-1}$. Hence the control weight for $\text{Co}(\text{L}^p(G))$ is given by*

$$w_0(x, h) = \max(\Delta_G(0, h)^{-1}, \Delta_G(0, h)) \preceq (|a| + |a|^{-1})^{|c-1|} .$$

Hence Theorem 3.4, together with the formula from 4.10 (with $d = 2, c = 1/2, u_1 = |c - 1| = 1/2, u_2 = s = 0$) yields that for $k \geq 105$, any function $\psi \in \text{L}^2(\mathbb{R}^2)$ with integrable derivatives of order up to $2k$ and vanishing moments of order k on the coordinate axis $\{0\} \times \mathbb{R}$ will be an atom for all coorbit spaces $\text{Co}(\text{L}^p(G))$, $1 \leq p \leq 2$. Thus a the shearlet analog of 1.2 follows from Theorem 2.7.

Remark 4.12. *For dimension $d = 2$, the examples covered in this section contain a system of representatives of admissible dilation groups up to conjugacy and finite index, see [13, 14]. It is easy to see that passing from an admissible dilation group H_1 to a conjugate group $H_2 = gH_1g^{-1}$, the open dual orbit of H_1 is strongly temperately embedded with respect to a weight w_1 on H_1 if and only if the open dual orbit H_2 is strongly temperately embedded with respect to the weight $w_2 : H_2 \ni h_2 \mapsto w_1(g^{-1}h_2g)$. Also, it is clear that the finite index condition has no effect on this property. Thus we have indeed verified that for all reasonable choices of weights on an admissible dilation group in dimension two, the condition of Theorem 3.4 is fulfilled.*

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